Multivariate models of equity returns for investment guarantees valuation

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This is mainly based upon the paper:

- "Multivariate models of equity returns for investment guarantees valuation"
- Written with Christian-Marc Panneton
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Earlier version of the paper has been presented at the Stochastic Modeling Symposium and Investment Seminar in Toronto in 2006.
About the authors

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Insurance market has seen the rise of products linked to the equity market through investment guarantees:

- Segregated funds / variable annuities;
- Equity-indexed annuities (EIA);

Due to the nature of the contract (put option combined with mortality), it is necessary to appropriately model the product and the underlying assets to compute sufficient provisions.

- Dynamic hedging may be difficult in certain cases;
- CTE provisions have become the standard;
CIA’s Task Force on Segregated Fund Investments in 2002:
- Recommended the use of stochastic equity returns models;
- No specific model has been mandated;
- Has to be at least as conservative as the CIA’s calibration points;

American Academy of Actuaries’ RBC C3 Phase II report
- Also recommended the use of stochastic equity returns models;
- Presented the stochastic log-volatility model;
We will take a look at:

- Concept of stochastic equity returns model
- Desired features
- Models proposed
- Valuation of investment guarantees
- Univariate vs multivariate
Stochastic equity returns model

- Model for the time-varying dynamics of the equity return
  - Random over time: stochastic process
  - Continuous-time vs discrete-time
  - Simple compounding vs continuous compounding

- Continuous compounding
  - Let $S_t$ be the price of a stock or the value of an equity index at time $t$
  - Then, the (continuously compounded) periodic return is

$$y_t = \log \left( \frac{S_t}{S_{t-1}} \right)$$

or

$$S_t = S_{t-1} \exp (y_t).$$
Known models

- Black-Scholes
  - Continuous-time model;
  - $S_t$ follows a geometric Brownian motion or $y_t$ follows an arithmetic Brownian motion;

- Independent log-normal (ILN)
  - Discrete-time equivalent of the geometric Brownian motion i.e.
    \[ y_t = \mu + \varepsilon_t \]
    where $\varepsilon_t \sim N(0, \sigma)$.

- Both are equivalent to saying returns follow a random walk
  - Cannot predict future value of the stock with historical data;
  - Fails in practice because of time-varying volatility: predictability in the squares of the returns;
Illustration of volatility

  - Daily returns, 60 trading days before, annualized
Models of heteroskedasticity

- Most famous are: ARCH of Engle (1982), GARCH (1,1) of Bollerslev (1986)
- More recent:
  - Regime-switching (RS) model (more details later): Hamilton (1989), Hardy (2001)
    - Combination of GARCH(1,1) and a regime-switching framework in Gray (1996)
    - Regime-switching model with a drawdown measure in Panneton (2002)
  - Stochastic volatility model (SV)
    - American Academy of Actuaries
    - Most important academic authors: Harvey, Ruiz, Shephard, Jacquier, etc.
    - Difficult to estimate: will not be further considered
- Idea: $\sigma_t^2$ is given by a model that is a function of past observations
Models of heteroskedasticity

- The main difference between classes of models is the dynamics of $\sigma_t^2$.
- In a ARCH/GARCH model, $\sigma_t^2$ is a deterministic function of past errors and volatilities
  - Fully predictable given past observations $y_{t-1}, y_{t-2}, \ldots$
- In a RS and SV model, $\sigma_t^2$ is a stochastic function of past errors and volatilities
  - Volatility is a stochastic process
  - Given past observations $y_{t-1}, y_{t-2}, \ldots$, the volatility is a random variable.
- Look at GARCH and RSLN models in more details.
In a GARCH model, we focus on the dynamics of $\sigma_t^2$.

Represented as

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

in a GARCH (1,1) model, where $\varepsilon_t = y_t - \mu$.

Suppose you know $\sigma_0$ and $y_0$. Then,

$$\varepsilon_0 = y_0 - \mu$$
$$\sigma_1^2 = \alpha_0 + \alpha_1 \varepsilon_0^2 + \beta_1 \sigma_0^2$$

and the rest of the computations are similar.

Consequently, $\sigma_t^2 \mid y_{t-1}, \ldots, y_0$ is deterministic.
Regime-switching models

- A regime-switching model is characterized by the addition of an unobserved process $\rho_t$ that takes $K$ different values, i.e. $\rho_t \in \{1, 2, ..., K\}$.
- In terms of stochastic processes, $\{\rho_t, t = 0, 1, 2, ...\}$ is a Markov chain.
- Dynamics of returns depend on the current state of $\rho_t$.
- In the RSLN model of Hardy (2001), the returns follow a different ILN model in each state:

$$ (y_t | \rho_t = k) \sim N(\mu_k, \sigma_k) $$

- It remains to determine the values of $\mu$ and $\sigma$ in the different regimes $(\mu_1, \mu_2, ..., \mu_K, \sigma_1, ..., \sigma_K)$ and the transition probabilities.
- With the dataset presented, RSLN models have both a better overall fit and better fit in the tails.
Value-at-Risk (VaR) is probably the most famous valuation metric.
VaR has some issues: CTE is better and recommended.
Mathematically,

\[ CTE_\alpha [X] = E[X | X > VaR_\alpha [X]]. \]

Intuitively, it is the mean loss given that losses are in the greatest 100 \((1 - \alpha)\)% of the sample.

Simple closed-form solution for RSLN models

In the context of the dataset used in Hardy (2001), it has been shown that:

- CTEs computed with regime-switching models are higher than with GARCH(1,1).
Introduction

- Previous models focused on the dynamics of one asset (univariate models);
- Segregated funds / variable annuities products are sold on portfolios of assets;
- Diversification and correlation (dependence) effects have to be appropriately represented;
- Purpose of what follows: present models of equity returns for multiple assets (multivariate models)
  - Understand dynamics of equity markets;
  - Valuation of investment guarantees (CTE);
Outline

1 Multivariate models
   - Important characteristics;
   - How do we build them;
   - Copulas (very briefly);
   - Multivariate extensions of known models;

2 Estimation
   - Maximum likelihood;

3 Valuation of investment guarantees
   - Closed-form solutions;
   - Simulation;

4 Numerical example
   - Dataset;
   - Comparison;
   - Validation;
   - Valuation of investment guarantees;
Characteristics

- What is the difference between modeling a single asset and a portfolio of assets?
  - **Important**: dependence relationship between equity returns
    - Strength of the relationship
    - How does it behave over time?
  - Each asset should have time-varying volatility as well;
  - All important issues that need to be addressed in a multivariate setting;
Characteristics

- Time-varying dependence relationship
  - Less obvious but intuitive
  - Increased dependence in times of crises
  - Multiple examples
    - Most obvious is the world we observe right now

- Time-varying volatility
  - Stochastic (Regime-switching, SV)
  - Predictable (GARCH processes)
  - Same idea in a multivariate framework
Time-varying correlation

- Following graph measures rolling correlation between S&P500 and S&P TSX from 2000 until Oct. 31st, 2008;
- Daily returns, 60 trading days before, annualized
Time-varying correlation

- Clearly, there is variation in the correlation over time;
  - Currently stronger than it were in 2005 before the onset of the subprime crisis.
- Will have an important impact on diversification.
  - Much more difficult to diversify during crises
- Backed by literature in financial empirical research: Tse (2000)
  - The author presents a test of time-varying correlation;
  - Shows statistically significant time-varying correlation in Asian equity markets;
  - Constant correlation in Asian futures and exchange rates markets.
Important features:
- Time-varying volatility and correlation
- Tractability
- Ease of application

Models considered:
- Multivariate regime-switching;
- Multivariate GARCH models;
- Copulas (very briefly)
Modeling dependence

2 approaches

1. Directly through the distribution of $\epsilon_t$
   - Joins markets through the multivariate distribution in itself
   - Multivariate normal distribution is common

2. Copulas
   - Joins markets through a function of each univariate c.d.f.
   - More difficult to account for time-varying dependence
   - Will briefly discuss this approach
Some notation

- Assume for the moment that we apply the first approach to dependence.

- Let

\[ \mathbf{y}_t \equiv \left[ y_{t}^{(1)}, \ldots, y_{t}^{(N)} \right] \]

represent the set of returns for each of the \( N \) assets of a portfolio.

- Dynamics of returns in general

\[ \mathbf{y}_t = f(\mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots) + \mathbf{\varepsilon}_t \]

where \( f \) is some function of previous returns and \( \mathbf{\varepsilon}_t \) is a vector of errors, which obeys some multivariate distribution.

- For simplicity, we assume \( f \) is some constant \( \mu \) which is the mean return.
How do we account for time-varying volatility and correlation?

Let

$$H_t \equiv \text{Var} \left[ \varepsilon_t \mid y_{t-1}, y_{t-2}, \ldots \right].$$

Multivariate GARCH models

- $H_t$ is predictable

Regime-switching models (and SVOL models)

- $H_t$ is random
- Once we know $y_{t-1}, y_{t-2}, \ldots$, the time $t$ covariance matrix is still random
- There is an unobservable process that affects the volatilities and correlations
Copulas

- 2nd approach for dependence modeling;
- What is a copula?

  - Approach that creates dependence between random variables using some function $C$ of the marginal c.d.f.
  - Assume for example the returns of each market $y_t^{(i)}$ follows an ILN model. Then, the c.d.f. of $y_t^{(i)}$ is $\Phi^{(i)}(.)$.
  - A copula would create a vector of r.v. with a joint c.d.f. $C \left( \Phi^{(1)}(.), ..., \Phi^{(N)}(.) \right)$.
  - $C$ is the copula and defines the dependence relationship between the random variables.
Copulas - Features

- Large literature on copulas allow the modeler to find a copula tailored to the type of dependence observed;
- Not more complicated to estimate the model or to simulate from;
- However, in the context of portfolios of more than 2 assets, choice of copulas is limited.
  - Gaussian or Student copulas (Elliptical copulas)
- Difficult to account for time-varying dependence
  - Not straightforward with a copula;
  - Possible and subject of future research;
- For the rest of the presentation:
  - Focus on first approach to dependence;
  - Focus on multivariate regime-switching and GARCH models;
Characteristics

- Very similar to the approach taken in the RSLN model;
- Extension to multivariate framework;
- 2 approaches:
  - Global regime;
  - Local regimes;
Global regime

- Unobserved process influences all markets at the same time;
- Markov process randomly moves from regime 1 (low vol.) to 2 (high vol.) over time;
- Once it is at regime 2 (1), all markets have higher (lower) volatilities for the same amount of time;
- Correlation also depends on the regime.
- Intuitively, higher (lower) correlation in the high (low) volatility regime.
- Realistic for crises but may ignore market specificities
- Parsimonious
Local regimes

- Basically add regimes to the global regime approach so that some markets may have different local dynamics;
  - Japan market may be in high vol. regime while U.S. market is in low vol. regime;
  - Some local events may have a short-term effect on returns that do not influence other markets;
- More realistic;
- Many more parameters to estimate;
- Comparison of both approaches later;
Global regime - Model

- Let $\rho_t$ denote the status of the unobserved global regime process (not to be confused with correlation).
- Then,
  $$y_t = \mu_{\rho_t} + \varepsilon_t$$

  where
  $$\varepsilon_t | \rho_t \sim \text{MVN} \left( 0, \Sigma_{\rho_t} \right).$$

- There are $K$ different regimes, that is $\rho_t \in \{1, 2, ..., K\}$. $K = 2$ or 3 is sufficient most of the time, even in a multivariate setting.
- This is the simplest formulation of the MRSLN model.
Global regime - Example

- Suppose there are two regimes: low (1) and high (2) volatility.
- Then, we have 2 states with probabilities of transition given by $p_{12}$ and $p_{21}$ (and their reciprocal).
- Assume at time $t$, we are in a high (2) volatility regime, i.e. $\rho_t = 2$. Then for this specific time period,
  \[ (\mathbf{y}_t | \rho_t = 2) \sim \text{MVN} (\mu_2, \Sigma_2). \]
- We require to estimate $\mu_2$ and $\Sigma_2$ which are a vector and matrix respectively.
- Similarly, if at time $t + 1$, we move back to the low volatility regime, then
  \[ (\mathbf{y}_{t+1} | \rho_{t+1} = 1) \sim \text{MVN} (\mu_1, \Sigma_1) \]
  with $\mu_1$ and $\Sigma_1$ estimated.
Local regimes - Illustration

- Suppose for the illustration that there are 2 assets (Canada, U.S.) and each have 2 regimes (low, high vol.)

- Then, it is equivalent to a global regime model with 4 regimes:
  - Canada low vol. - U.S. low vol: 11
  - Canada low vol. - U.S. high vol: 12
  - Canada high vol. - U.S. low vol: 21
  - Canada high vol. - U.S. high vol: 22

- We have 4 states with 12 different probabilities of transition.
- Volatilities, correlations and mean returns are also necessary for each of the 4 regimes.
- Very important number of parameters, even for 2 assets only.
- More parsimonious to add a third regime for a market that is suspected to have important local dynamics.
Introduction

- We have previously reviewed the univariate GARCH model.
- Idea is similar in a multivariate setting:
  - Current covariance matrix is a function of past errors and covariances.
- Univariate models may also show useful for some classes of multivariate models.
- 4 models mainly:
  - Vech;
  - BEKK;
  - DCC;
  - CCORR;
Issues

- Parsimony is important for both classes of models but especially GARCH models;
- Positive (semi-) definiteness of covariance matrix
  - Suppose we have a portfolio defined by the weights $w$.
  - Then, it is essential that

$$\text{Var} \left[ w^\top y_t \right] = \text{Var} \left[ \sum_{i=1}^{N} w^{(i)} y^{(i)}_t \right]$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} w^{(i)} w^{(j)} H_{t}^{(i,j)}$$

$$\geq 0.$$  

- A matrix will not be PSD if $\text{Var} \left[ w^\top y_t \right] < 0$.
- More important with GARCH models.
VECH model

- Presented by Bollerslev, Engle and Wooldridge (1988) as an approach to model the time-varying market premium in the CAPM;
- Focus on the simplest representation: diagonal VECH;
- Essentially a GARCH(1,1) on each element of $H_t$;
  - This formulation does not guarantee PSD;
  - Has to use other simple formulation;
- Important feature: volatility of one market is NOT a function of the volatility of other markets;
VECH model

- Reformulate instead as

\[
H_t = \left( CC^T \right) + \left( AA^T \right) \otimes \left( \varepsilon_{t-1} \varepsilon_{t-1}^T \right) + \left( BB^T \right) \otimes H_{t-1}
\]

with \( \otimes \) being the element-by-element multiplication.

- This guarantees PSD.

- **Simplifications**: \( A, B \) and \( C \) can be vectors or scalars.
BEKK model

- Formulation that guarantees PSD.
- Features:
  - Volatility of one market is a function of the volatility of other markets.
  - A lot more parameters to estimate.
- BEKK model is

\[
H_t = C^T C + A \left( \varepsilon_{t-1} \varepsilon_{t-1}^T \right) A^T + BH_{t-1}B^T
\]

where \( A, B \) and \( C \) are full matrices of coefficients to estimate.

- **Simplifications**: \( A, B \) and \( C \) can be vectors or scalars.
DCC model

- Introduced by Engle (2002): Dynamic Conditional Correlation
- **Idea**: correlation of returns is equivalent to covariance of **standardized** returns (with variance 1)
- Model is built in 2 steps:
  - Volatility dynamics;
  - Correlation dynamics;
- We can always write the dynamics of the covariance matrix as
  \[ H_t = D_t R_t D_t. \]
DCC model

- Estimated in two steps also:
  - Volatility dynamics for each market;
  - Given the residuals for each market, the dynamics of correlation are estimated;
- Engle presents correlation processes that use 1 or 2 parameters;
- Very parsimonious;
Constant correlation model (CCORR)

- Introduced by Bollerslev (1990);
- Special (important) case of the DCC model:

\[ H_t = D_t R D_t \]

where \( R \) is the constant correlation matrix.

- Can also be estimated in two steps:
  - Volatility dynamics
  - Correlation dynamics
Maximum likelihood estimation

- Usual approach;
- Find parameters that maximize probability of observing the sample;
- Multivariate regime-switching models
- Multivariate GARCH models
  - VECH and BEKK models
  - DCC and CCORR models
Multivariate regime-switching models

- As simple as in a univariate framework.
  - Instead we use the multivariate p.d.f. of $\mathbf{y}_t \mid \mathbf{y}_{t-1}, \ldots, \mathbf{y}_0$
- Can be done in Excel in combination with the Solver.
Multivariate GARCH models

- We can estimate all parameters at once or use a two-step approach.

  All parameters at once (All models):
  
  - $y_t | y_{t-1}, ..., y_0$ is a $N$-variate normal distribution with mean $\mu$ and covariance $H_t$.
  
  - Maximize likelihood function using the p.d.f. of a $N$-variate normal variable.

  Two-step estimation (DCC and CCORR only):
  
  - Perform $N$ univariate estimations of the volatility dynamics, one for each market.
  
  - Compute the resulting standardized residuals.
  
  - Find the 1 or 2 parameters that define the correlation dynamics in the DCC model.
We define the portfolio weights $\mathbf{w}$.

The portfolio return is

$$y_t^{(P)} = \mathbf{w}^\top \mathbf{y}_t$$

$$= \sum_{i=1}^{N} w^{(i)} y^{(i)}_t.$$ 

We are looking for the amount of money that would cover the mean losses over the Value-at-Risk.
Given that we know the regime or the number of visits to each regime, then

\[ y_t^{(P)} \mid \rho_t \]

has a normal distribution.

Use Hardy (2001, 2003) results with respect to the CTE with one asset with the modification that

\[
\begin{align*}
\mu^* (k) &= kw'\mu_1 + (n - k) w'\mu_2 \\
\sigma^* (k) &= \sqrt{kw'\Sigma_1 w + (n - k) w'\Sigma_2 w}.
\end{align*}
\]
No closed-form solution since

\[ y_{t+n}^{(P)} \mid y_t, y_{t-1}, \ldots, n > 1 \]

is not normal anymore.

Solutions:

- Approximation;
- Naive simulation;
- Simulation with control variate;
Approximation: assume that

$$\text{Var} \left[ y_{t+n}^{(P)} \right] \equiv \mathbf{w}^\top E \left[ H_{t+n} | y_t, y_{t-1}, \ldots \right] \mathbf{w}, \quad n > 1$$

- CTE results from a lognormal distribution with independent but not identically distributed returns: closed-form solution exists.
- $E \left[ H_{t+n} | y_t, y_{t-1}, \ldots \right], \quad n > 1$ is simple to derive

Naive simulation:
- Simulate multiple scenarios of returns for all assets over the time period considered;
- Compute the loss for each scenario;
- Compute a realization of the CTE as the mean of the 5% largest losses;
- Repeat if necessary;

Simulation with control variate
- Use the previous approximated CTE value to reduce the number of necessary scenarios or to increase precision in the CTE value.
Introduction

• Heart of the analysis;

• Outline:
  • Dataset;
  • Global vs local regimes;
  • Model selection;
  • Model validation;
  • CTE;
Dataset

- 4 markets: Canada, U.S., United Kingdom, Japan
- Indices:
  - Canada: S&P TSX total return index
  - U.S.: S&P 500 total return index
  - Japan: monthly historical return of TOPIX
Global vs local regimes

- 2 pairs of markets:
  - Canada and U.S.;
  - U.S. and Japan;

- Intuitively, global regime should work better than local regimes for highly correlated markets.

Local regimes:

- Dependent: each probability of transition $p_{11,12}$ is estimated
- Independent:
  - Simplification of the dependent approach;
  - Probability of transition $p_{11,12}$ is the product of marginal transition probabilities;

- Compare fit on the basis of penalized-likelihood criteria
  - The more parameters a model have, the better the fit should be
Global vs local regimes

- # is the number of parameters, AIC and SBC are the Akaike and Schwartz-Bayes Criteria
- Global regime does well in both cases

<table>
<thead>
<tr>
<th>Model</th>
<th>#</th>
<th>AIC</th>
<th>SBC</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. and Canada</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Global</td>
<td>12</td>
<td>2370.90</td>
<td>2344.56</td>
</tr>
<tr>
<td>Indep. local</td>
<td>16</td>
<td>2355.46</td>
<td>2320.34</td>
</tr>
<tr>
<td>Dep. local</td>
<td>28</td>
<td>2356.46</td>
<td>2295.00</td>
</tr>
<tr>
<td>U.S. and Japan</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Global</td>
<td>12</td>
<td>2040.01</td>
<td>2013.67</td>
</tr>
<tr>
<td>Indep. local</td>
<td>16</td>
<td>2043.55</td>
<td>2008.43</td>
</tr>
<tr>
<td>Dep. local</td>
<td>28</td>
<td>2033.64</td>
<td>1972.18</td>
</tr>
</tbody>
</table>
Model selection

- Choose most parsimonious models among a set of regime-switching and GARCH models.
  - More parameters should mean better fit. How much better is the fit?
- Penalized-likelihood criteria
  - LRT is the likelihood ratio test. Checks statistically the significance of the added parameters in nested models.

**Regime-switching models**

<table>
<thead>
<tr>
<th>Model</th>
<th>#</th>
<th>AIC</th>
<th>SBC</th>
<th>LRT</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRSLN(2,1)</td>
<td>24</td>
<td>4399,72</td>
<td>4347,04</td>
<td>n/a</td>
</tr>
<tr>
<td>MRSLN(2,2)</td>
<td>30</td>
<td>4396,64</td>
<td>4330,79</td>
<td>0,4414</td>
</tr>
<tr>
<td>MRSLN(3,1)</td>
<td>36</td>
<td>4421,81</td>
<td>4342,79</td>
<td>&lt; 10^{-7}</td>
</tr>
<tr>
<td>MRSLN(3,3)</td>
<td>48</td>
<td>4415,14</td>
<td>4309,77</td>
<td>&lt; 10^{-7}</td>
</tr>
</tbody>
</table>

- Most parsimonious models are MRSLN(2,1) and MRSLN(3,1).
  - One correlation matrix
  - 2 or 3 regimes
Model selection

- **GARCH models**

<table>
<thead>
<tr>
<th>Model</th>
<th>#</th>
<th>AIC</th>
<th>SBC</th>
</tr>
</thead>
<tbody>
<tr>
<td>VECH</td>
<td>34</td>
<td>4389.36</td>
<td>4314.73</td>
</tr>
<tr>
<td>Vector VECH</td>
<td>22</td>
<td><strong>4402.74</strong></td>
<td>4354.45</td>
</tr>
<tr>
<td>Scalar VECH</td>
<td>16</td>
<td>4399.59</td>
<td><strong>4364.47</strong></td>
</tr>
<tr>
<td>CCORR</td>
<td>22</td>
<td>4402.38</td>
<td>4354.08</td>
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<tr>
<td>DCC-INT</td>
<td>23</td>
<td><strong>4410.46</strong></td>
<td>4359.98</td>
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<tr>
<td>DCC-MR</td>
<td>24</td>
<td><strong>4410.13</strong></td>
<td>4357.45</td>
</tr>
</tbody>
</table>

- **Results:**
  - BEKK model is clearly over-parameterized (not in table)
  - Most parsimonious models are the DCC models, Vector VECH and constant correlation.
  - Overall, based on parsimony criteria, GARCH models have a better global fit.
Validation tests

- Parsimony criteria compare models and tell which one is the most parsimonious;
  - Best fit taking into account the number of parameters;
- Selection tests tell nothing about the quality of the fit itself
  - Danger: fit can be bad for all models and one model can be just slightly better;
- 2 sets of tests:
  - Normality tests;
  - Heteroskedasticity tests;
Validation tests

- Normality tests: residuals of a well-fitted model should be approximately normal
  - Models that lack fit in the tails will most likely fail those tests;
  - Jarque-Bera and Shapiro-Wilk tests;
  - Q-Q plots;

- Heteroskedasticity tests
  - Heteroskedasticity: time-varying variance and covariance;
  - A well-fitted model in covariance should no longer show signs of heteroskedasticity in its residuals;
  - ARCH, Hosking and Engle tests;
Normality tests

- Both Jarque-Bera and Shapiro-Wilk tests show that most GARCH models fail normality tests.
- Lack of fit in the tails is shown in the following graph (constant correlation GARCH)
The following graph holds for the MRSLN(2,1) model.
Heteroskedasticity tests

- Show the results of Hosking test with 6 lags (similar with 12 lags)
- Conclusions inferred from the other 2 tests are similar.

<table>
<thead>
<tr>
<th>Model</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRSLN(2,1)</td>
<td>$&lt;10^{-4}$</td>
</tr>
<tr>
<td>MRSLN(3,1)</td>
<td>0.0088</td>
</tr>
<tr>
<td>MRSLN(3,3)</td>
<td>0.0059</td>
</tr>
<tr>
<td>CCORR</td>
<td>0.7178</td>
</tr>
<tr>
<td>Vector VECH</td>
<td>0.0038</td>
</tr>
<tr>
<td>DCC-INT</td>
<td>0.2203</td>
</tr>
<tr>
<td>BEKK</td>
<td>0.0075</td>
</tr>
</tbody>
</table>

- Two parsimonious GARCH models do not fail Hosking’s test
  - This is contrary to regime-switching models
  - Both complex models (MRSLN(3,3) and BEKK) also fail the test. Complexity does not result in quality of fit.
Summary of tests

- **Selection tests:**
  - Multivariate GARCH models had a better overall fit;

- **Normality tests:**
  - Regime-switching models had a better fit in the tails;

- **Heteroskedasticity tests:**
  - Multivariate GARCH models had a better overall fit;

Since dynamics of covariance account for a larger part of the sample than the tails, GARCH models proved to be better overall.

However, model selection should depend on what matters from the model: overall vs fit in the tails.
CTE provisions over one market

- We have computed 10-year CTEs, without discounting, with a 3% annual MER.
- GARCH CTEs have been computed with 10000 replications.
- 1st table: 100% allocation in each of the 4 markets

<table>
<thead>
<tr>
<th>CTE</th>
<th>S&amp;P TSX</th>
<th>S&amp;P 500</th>
<th>Topix</th>
<th>FTSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRSLN(2,1)</td>
<td>43.90%</td>
<td>32.00%</td>
<td>54.00%</td>
<td>56.20%</td>
</tr>
<tr>
<td>MRSLN(3,1)</td>
<td>47.70%</td>
<td>34.70%</td>
<td>54.50%</td>
<td>58.80%</td>
</tr>
<tr>
<td>MRSLN(3,3)</td>
<td>48.80%</td>
<td>35.90%</td>
<td>54.80%</td>
<td>60.40%</td>
</tr>
<tr>
<td>CCORR</td>
<td>32.88%</td>
<td>24.67%</td>
<td>51.51%</td>
<td>55.70%</td>
</tr>
<tr>
<td>DCC-INT</td>
<td>35.07%</td>
<td>23.55%</td>
<td>53.54%</td>
<td>53.82%</td>
</tr>
<tr>
<td>BEKK</td>
<td>29.71%</td>
<td>22.33%</td>
<td>55.91%</td>
<td>60.18%</td>
</tr>
</tbody>
</table>

- CTEs are higher with regime-switching models, especially for North American markets
CTE provisions over portfolios

- 2nd table: Different allocations in the 4 markets

<table>
<thead>
<tr>
<th>CTE</th>
<th>Ptf # 1</th>
<th>Ptf # 2</th>
<th>Ptf # 3</th>
<th>Ptf # 4</th>
<th>Ptf # 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRSLN(2,1)</td>
<td>35.20%</td>
<td>38.60%</td>
<td>33.00%</td>
<td>36.80%</td>
<td>33.80%</td>
</tr>
<tr>
<td>MRSLN(3,1)</td>
<td>38.70%</td>
<td>41.80%</td>
<td>34.30%</td>
<td>40.50%</td>
<td>36.80%</td>
</tr>
<tr>
<td>MRSLN(3,3)</td>
<td>37.10%</td>
<td>42.30%</td>
<td>32.10%</td>
<td>40.10%</td>
<td>35.10%</td>
</tr>
<tr>
<td>CCORR</td>
<td>24.70%</td>
<td>33.00%</td>
<td>27.28%</td>
<td>27.61%</td>
<td>25.18%</td>
</tr>
<tr>
<td>DCC-INT</td>
<td>24.97%</td>
<td>32.53%</td>
<td>25.80%</td>
<td>26.77%</td>
<td>25.97%</td>
</tr>
<tr>
<td>BEKK</td>
<td>21.84%</td>
<td>34.00%</td>
<td>26.69%</td>
<td>26.91%</td>
<td>24.94%</td>
</tr>
</tbody>
</table>

- CTEs are higher with regime-switching models, especially for portfolios with higher allocations in North American markets
Conclusion

- Given the results obtained with this specific dataset:
  - Cannot recommend a specific model for all purposes;
  - Depends on how the model will be used;
- Trade-off:
  - Fat tails (tail or local fit);
  - Heteroskedasticity in the covariances (overall or global fit);
  - None of the models presented do good in both areas;
- Recommendations depend on what matters:
  - Overall fit (explaining returns for example): multivariate GARCH;
  - Fit in the tails (CTE provisions): multivariate regime-switching;
Main reference:

- Boudreault, Mathieu and Christian-Marc Panneton (2008), "Multivariate models of equity returns for investment guarantees valuation", Accepted for publication in the North American Actuarial Journal

Important references:

Bibliography

- Other references:

Other references: