Stochastic Volatility Models: Calibrating, Pricing and Hedging

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Outline of the Presentation:

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- The Heston Model
- Price of Call Options
- Hedging Derivatives

Calibrating the Models
- Calibrating Black-Scholes
- Calibrating Heston

Hedging the Financial Guarantee
- Overview of Equity-Indexed Annuities
- Hedging the Financial Guarantee
- Numerical Example
- Extension to other payoffs and products

Conclusion
General Market Assumptions

- No transaction costs or taxes.
- Infinitely divisible securities.
- Anyone can borrow or lend at a constant risk-free rate $r$.
- No restrictions on short selling.
- No possibility of arbitrage.
The Black-Scholes Model

- Introduced by Black and Scholes (1973).
- Stock index price dynamics given by

\[
dS_t = \mu_{BS} S_t dt + \sigma_{BS} S_t dZ_t \quad t > 0, \\
S_0 = s
\]

where $\mu_{BS}$ and $\sigma_{BS}$ are constants and $Z_t$ is a standard Brownian motion.

- $\ln \left( \frac{S_t}{S_0} \right) \sim \text{Normal} \left( (\mu_{BS} - \frac{\sigma_{BS}^2}{2}) t, \sigma_{BS}^2 t \right)$
Two Measures

- Objective measure: historical prices.
- Risk-neutral measure:
  - Used to price derivatives.
  - Discounted price processes are martingales.
- Under the risk-neutral measure in Black-Scholes:

\[ dS_t = r S_t dt + \sigma_{BS} S_t d\tilde{Z}_t \]
Empirical and Black-Scholes Prices
Problems with Black-Scholes

- Does not fit the empirical distribution
  - High peaks
  - Fat tails
- Volatility smile
Dynamics of index price:

\[ dS_t = \mu S_t dt + \sqrt{v_t} S_t dZ_t^{(1)}, \]
\[ dv_t = a(t, v_t) dt + b(t, v_t) dZ_t^{(2)}, \]

Examples: SABR, CEV, 3/2, Heston, etc.
The Heston Model

- Introduced by Heston (1993).
- Dynamics of index price (objective measure):

\[
\begin{align*}
    dS_t &= \mu S_t \, dt + \sqrt{v_t} S_t \, dZ_t^{(1)}, \\
    dv_t &= \kappa' (\theta' - v_t) \, dt + \sigma \sqrt{v_t} \, dZ_t^{(2)},
\end{align*}
\]

(2)

where \( \mu, \kappa', \theta', \sigma \) are constants and \( \langle dZ_t^{(1)} \, dZ_t^{(2)} \rangle = \rho \, dt \).
Advantages of the Heston Model

- Stochastic volatility term: easier to fit the empirical distribution of log-returns.
- CIR process:
  - Cannot reach 0 under certain conditions.
  - Mean reversion.
  - Analytic forms for the price of European options.
- Popular in the industry.
Risk-Neutral Heston Parameters

- Dynamics have the same form under both measures
- Risk-neutral parameters:
  \[ \kappa = \kappa' + \lambda \]
  \[ \theta = \frac{\kappa' \theta'}{\kappa' + \lambda} \]
- Market price of volatility risk: \( \lambda v_t \).
Simulated Heston Prices

Figure: Weekly log-returns simulated using the Heston model, $\kappa = 5$, $\theta = 0.0178$, $\sigma = 0.1309$, $v_0 = 0.0286$, $\rho = -0.7025$. 

Stochastic Volatility Models
Financial Models
The Heston Model
Price of a European Call Option - Black-Scholes

\[ C^{BS}(S_t, K, \tau) = S_t \Phi(d_1) - K e^{-r(\tau)} \Phi(d_2), \]

where \( \Phi \) is the standard normal CDF, \( K \) is the strike price, \( \tau = T - t \) and

\[
\begin{align*}
    d_1 &= \frac{\log \frac{S_t}{K} + \left( r + \frac{\sigma_{BS}^2}{2} \right)(\tau)}{\sigma_{BS} \sqrt{\tau}} \\
    d_2 &= d_1 - \sigma_{BS} \sqrt{\tau},
\end{align*}
\]
Price of a European Call Option - Heston

\[ C^H(x_t, \nu_t, \tau) = K e^{-r\tau} (e^{x_t} P_1(x_t, \nu_t, \tau) - P_0(x_t, \nu_t, \tau)) \]

where \( x_t = \log\left( \frac{e^{r(T-t)} S_t}{K} \right) \), \( K \) is the strike price, \( \tau = T - t \) and

\[ P_j(x_t, \nu_t, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{\exp(iux_t + C_j(u, \tau) \theta + D_j(u, \tau) \nu_t)}{iu} \right) du, \]

for \( j = 0, 1 \).
The Greeks are sensitivities...

- $\Delta$: to changes in the price of the index.
- $\Gamma$: of the Delta to changes in the price of the index.
- $\vee$: to changes in the volatility.
Dynamic Hedging Using the Greeks

- Build a replicating portfolio with the premium obtained from the sale.
- Replicating portfolio meant to keep the value of the option at all time.
- Match the Greeks of the replicating portfolio and the derivative so they vary in similar ways.
Delta Hedging

- Protects against small changes in index prices.
- Replicating portfolio invested in the index and in the money market.
- We want

\[
\frac{\partial H}{\partial S_t} = \frac{\partial V}{\partial S_t}
\]

\[
\Leftrightarrow
\]

\[
\frac{\partial}{\partial S_t} (\alpha S_t + (1 - \alpha)) = \frac{\partial V}{\partial S_t}
\]

- So proportion of the portfolio invested in the index given by \( \Delta V,t \)
- Strategy is self-financing when applied in continuous time.
Gamma Hedging

- Improves the delta hedging strategy in discrete time.
- Replicating portfolio invested in the index, the money market and a derivative (option).
- We want \( \frac{\partial H}{\partial S_t} = \frac{\partial V}{\partial S_t} \) and \( \frac{\partial^2 H}{\partial S_t^2} = \frac{\partial^2 V}{\partial S_t^2} \).
- Proportion of the portfolio invested in the option given by \( \frac{\Gamma_{V,t}}{\Gamma_{C,t}} \).
- Proportion of the portfolio invested in the index given by \( \Delta_{V,t} - \frac{\Gamma_{V,t}}{\Gamma_{C,t}} \Delta_{C,t} \).
Vega Hedging

- Protects the insurer against small changes in index prices and volatility.
- Replicating portfolio invested in the index, the money market and a derivative.
- We want \( \frac{\partial H}{\partial S_t} = \frac{\partial V}{\partial S_t} \) and \( \frac{\partial H}{\partial \nu_t} = \frac{\partial V}{\partial \nu_t} \).
- Proportion of the portfolio invested in the option given by
  \[
  \frac{\nu_{V,t}}{\nu_{C,t}}
  \]
- Proportion of the portfolio invested in the index given by
  \[
  \Delta V_{t} - \frac{\nu_{V,t}}{\nu_{C,t}} \Delta C_{t}
  \]
Hedging Errors

- Due to the discretization of the hedging process.
- Occur when rebalancing the replicating portfolio.
- Hedging error at time $t$ defined by

$$HE_t = V_t - H_t.$$ 

- Total discounted hedging error given by

$$PV(HE) = \sum_{j=1}^{mT} e^{-jr} HE_j.$$
Calibration

- Assumption: market is efficient and contains all available information
- Two types of measure to calibrate:
  - Objective measure: historical returns
  - Risk-neutral measure: option prices
Calibrating the physical measure - Black-Scholes

- Use maximum likelihood estimators:

\[
\hat{\sigma}^2_{BS} = \frac{1}{n-1} \sum_{i=1}^{n} (r_i - \bar{r})^2
\]

\[
\hat{\mu}_{BS} = \bar{r} + 0.5 \hat{\sigma}^2_{BS}
\]

with \( \bar{r} = \frac{1}{n} \sum_{i=1}^{n} r_i \).

- With S&P500 weekly data from May 20, 1996 to May 16, 2011, we get \( \hat{\mu}_{BS} = 0.0637 \) and \( \hat{\sigma}_{BS} = 0.19 \).
Calibrating the risk-neutral measure - Black-Scholes

- Only parameter to calibrate is the risk-free rate.
- Can be observed on the market.
- Use longer maturities since EIAs are long-term.
- 1- to 10-year range from 0.18% to 3.15%: choose 2.00%.
- Improvement: use deterministic yield curve.
Calibrating the risk-neutral measure - Heston

- Idea: match market prices of European call options.
- Minimize the square of the difference between model and market prices.
- Can use weights based on liquidity of call.
- Obtain a function to optimize (use Matlab).
- Check that, on average, model prices are between the bid and the ask prices.
Calibrating the full Heston model

- Harder to calibrate since volatility term $v_t$ is not directly observable.
- Calibrating both measures simultaneously requires extensive intraday quotes and historical option prices.
- Proposed methods:
  - Maximum likelihood (Aït-Sahalia and Kimmel (2007))
  - Method of moments (Garcia et al. (2011))
  - Generalized method of moments (Bollerslev et al. (2011))
Our calibration of Heston

- First calibrated risk-neutral measure.
- Parameters obtained: $\kappa = 5.1793$, $\theta = 0.0178$, $\sigma = 0.1309$, $v_0 = 0.0286$, $\rho = -0.7025$.
- Will test different volatility risk premia.
- To match historical data, need $\lambda = 2.62$. 
Equity-Indexed Annuities in the Literature

- First studied under the Black-Scholes model by Brennan and Schwartz (1976) and Boyle and Schwartz (1977)
- Hardy (2003) discusses product design and pricing techniques
- Tiong (2000) and Lee (2003) present closed-form expressions for the price of the financial guarantees embedded in EIAs
- Lin and Tan (2003) price EIAs under stochastic interest rate models
- Lin et al. (2009) use a regime-switching model to value EIAs
Financial Guarantee in EIAs

- Maturity around 5 to 15 years.
- Guaranteed return on initial investment.
- Additional return based on the performance of a stock index.
- Additional return may be reduced or capped.
- Actual return of the EIA depends on its design (point-to-point, annual reset, ...).
Point-to-Point Payoff

Payoff based on the value of the index at inception and at maturity of the contract:

\[ B^{PTP}(S_T, T) = \max \left( 1 + \alpha \left( \frac{S_T}{S_0} - 1 \right), \varrho (1 + g)^T \right), \]  

where:

- \( \alpha \): participation in index return,
- \( \varrho \) is proportion of initial investment that is guaranteed.
Pricing Point-to-Point EIAs

- Can re-write (3) as:

\[ B^{PTP}(S_T, T) = K + \frac{\alpha}{S_0} \max (S_T - L, 0), \]

where \( K = \varrho (1 + g)^T \) and \( L = S_0 \left( \frac{K - 1 + \alpha}{\alpha} \right) \).

- Price at time \( t \) of EIA with maturity \( T \) is

\[ P_t(S_t, \tau) = Ke^{-r\tau} + \frac{\alpha}{S_0} C(S_t, L, \tau), \]
Assessing the Performance of the Hedge

1. Start with index price $S_0$ at inception of the contract.
2. Simulate $S_{\delta t}$ at each time step using Heston model.
3. Given $S_{\delta t}$, calculate hedging error.
4. At the end, discount hedging error back to $t = 0$.
5. Repeat steps 1 to 4 100,000 times to obtain distribution of total hedging error.
Simulating Heston Prices

- Simple Euler discretization may simulate negative volatility.
- Need to go one order higher in Itô-Taylor expansion: Milstein discretization (see Kloeden and Platen (1992))

\[ v_{t+\delta} = v_t + (\kappa'(\theta' - v_t) - \frac{1}{2}\sigma^2)\delta + \sigma \sqrt{v_t \delta} N(0, 1) + \frac{1}{2} \sigma^2 \delta (N(0, 1))^2. \]  

(4)

- If \( \frac{4\kappa\theta}{\sigma^2} > 1 \), \( v_{t+\delta} \) should not become negative.
- Absorption assumption: let \( v_{t+\delta} = \max(0, v'_{t+\delta}) \), where \( v'_{t+\delta} \) is obtained using (4).
Numerical Example - Assumptions

- 10-year maturity point-to-point EIA with $g = 0$ and $\varrho = 1$.
- Participation rate $\alpha$ chosen so that the price of the EIA is 1.
- Risk-free rate $r = 0.02$.
- Index prices follow Heston model with different volatility risk premia $\lambda$. 
Stochastic Volatility Models

Hedging the Financial Guarantee

Numerical Example

Black-Scholes Delta Hedging

Figure: Present values of hedging errors resulting from a Black-Scholes delta hedging strategy for different values of $\lambda$, $\alpha = 0.5723$
Black-Scholes Gamma Hedging

(a) $\lambda = -1$

(b) $\lambda = 0$

(c) $\lambda = 2.62$

**Figure:** Present values of hedging errors resulting from a Black-Scholes gamma hedging strategy for different values of $\lambda$, $\alpha = 0.5723$
Heston Delta Hedging

(a) $\lambda = -1$

(b) $\lambda = 0$

(c) $\lambda = 2.62$

Figure: Present values of hedging errors resulting from a Heston delta hedging strategy for different values of $\lambda$, $\alpha = 0.6961$
Heston Gamma Hedging

Figure: Present values of hedging errors resulting from a Heston gamma hedging strategy for different values of $\lambda$, $\alpha = 0.6961$
Heston Vega Hedging

Figure: Present values of hedging errors resulting from a Heston gamma hedging strategy for different values of $\lambda$, $\alpha = 0.6961$
More Complex Payoffs

- Other possible payoffs: annual ratchet, high water mark, etc.
- Harder to find closed-form expressions in stochastic volatility models.
- Nested Monte-Carlo simulations: possible, but computationally intensive.
- Sensitivity of payoffs to stochastic volatility.
Variable Annuities

- GMMB: Put option on the fund value.
- Fund value is typically not an index.
- Volatility of fund value still a concern.
- GMMB more valuable when fund is more volatile.
Conclusion

- Stochastic volatility models:
  - Better fit for the heavy tails and high peaks of empirical distribution of log-returns.
  - Heston model allows for closed-form expressions.
- Stochastic volatility affects performance of hedge: Black-Scholes is not enough.
- Need to pay attention to volatility changes when pricing and hedging.


Thank you for your attention. Questions?